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ON $(\Delta^m, I)-$ LACUNARY STATISTICAL CONVERGENCE OF ORDER α

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Abstract. In this study, using the generalized difference operator Δ^m, we introduce the concepts of $(\Delta^m, I)-$lacunary statistical convergence of order α and lacunary strong Δ_p^m-summability of order α of sequences and give some relations about these concepts.

1. Introduction

In 1951, Fast [10] introduced the notion of statistical convergence and Schoenberg [18] reintroduced independently in 1959. Later on Çolak [2], Fridy [11], Šalát [19], Tripathy [23] and another researchers have studied the concept from the sequence space point of view and linked with the Summability theory.

The notion of $I-$convergence is a generalization of the statistical convergence. Kostyrko, Šalát and Wilczyński [15] introduced the notion of $I-$convergence. Some further results connected with the notion of $I-$convergence can be found in ([4],[5], [9],[16],[20],[21]).

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Throught this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Recently lacunary sequences have been studied in ([1],[3],[8],[12],[13],[22]).

A non-empty family $I \subseteq 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if $\phi \in I$, A, $B \in I$ implies $A \cup B \in I$ and $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family $F \subseteq 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if $\phi \notin F$, $A, B \in F$ implies $A \cap B \in F$ and $A \in F$, $A \subset B$ implies $B \in F$.

If I is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets

$$F(I) = \{M \subset \mathbb{N} : \exists A \in I : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$.

A proper ideal I is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Throughout this study, I will stand for a proper admissible ideal of $\mathbb{N}$ and by a sequence we always mean a sequence of real numbers.

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The notion of difference sequence spaces was introduced by Kızmaz [14] and it was generalized by Et et al. ([6],[7],[9],[17],[21]) such as

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = \left(\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}\right)$, and so $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. If $x \in \Delta^m(X)$ then there exists one and only one $y = (y_k) \in X$ such that

$$x_k = \sum_{i=1}^{k-m} (-1)^m \binom{k-i-1}{m-1} y_i = \sum_{i=1}^{k} (-1)^m \binom{k+m-i-1}{m-1} y_{i-m},$$

$y_{1-m} = y_{2-m} = ... = y_0 = 0$, for sufficiently large k; for example, $k > 2m$. We use this truth to define in sequences $(2.1), (2.2)$ and (2.3).

2. Main Results

In this section, we describe the concepts of $(\Delta^m, I)-$lacunary statistical convergence of order α and lacunary strong Δ_p^m-summability of order α of sequences and give some relations about these concepts.

Definition 2.1. *Let $\theta = (k_r)$ be a lacunary sequence and $\alpha \in (0,1]$ be a fixed real number. We say that the sequence $x = (x_k)$ is $S_\theta^\alpha(\Delta^m, I)-$convergent (or $(\Delta^m, I)-$lacunary statistically convergent sequences of order α) if there is a real number L such that*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in I,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denote the αth power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ..., h_r^\alpha, ...)$. In this case we write $S_\theta^\alpha(\Delta^m, I) - \lim x_k = L$ or $x_k \to L(S_\theta^\alpha(\Delta^m, I))$. We will denote the set of all $S_\theta^\alpha(\Delta^m, I)-$convergent sequences by $S_\theta^\alpha(\Delta^m, I)$. If $\theta = (2^r)$, then we will write $S^\alpha(\Delta^m, I)$ in the place of $S_\theta^\alpha(\Delta^m, I)$ and if $\alpha = 1$ and $\theta = (2^r)$, then we will write $S(\Delta^m, I)$ in the place of $S_\theta^\alpha(\Delta^m, I)$.

Definition 2.2. *Let $\theta = (k_r)$ be a lacunary sequence and $\alpha \in (0,1]$ be a fixed real number. We say that the sequence $x = (x_k)$ is $N_\theta^\alpha(\Delta^m, I)-$summable to L (or lacunary strongly Δ^m-summable sequence of order α) if, for any $\varepsilon > 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |\Delta^m x_k - L| \geq \varepsilon\right\} \in I.$$

In this case we write $x_k \to L(N_\theta^\alpha(\Delta^m, I))$ and we will denote the set of all $N_\theta^\alpha(\Delta^m, I)-$summable sequences by $N_\theta^\alpha(\Delta^m, I)$.

It can be shown that $S_\theta^\alpha(\Delta^m, I)-$ convergence is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general.

The inclusion parts of the following three theorems are straightforward, so we omit these parts of their proofs.

Theorem 2.1. *If $x_k \to L(S_\theta^\alpha(\Delta^m, I))$, then $x_k \to L(S_\theta^\beta(\Delta^m, I))$ and the inclusion is proper.*

Proof. Define a sequence $x=(x_k)$ by

$$\Delta^m x_k = \begin{cases} k, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases}. \tag{2.1}$$

Then $x \in S_\theta^\beta(\Delta^m, I)$ for $\frac{1}{2} < \beta \le 1$ but $x \notin S_\theta^\alpha(\Delta^m, I)$ for $0 < \alpha \le \frac{1}{2}$. □

Theorem 2.2. *If $x_k \to L(N_\theta^\alpha(\Delta^m, I))$, then $x_k \to L(N_\theta^\beta(\Delta^m, I))$ and the inclusion is proper.*

Proof. Define a sequence $x=(x_k)$ by

$$\Delta^m x_k = \begin{cases} 1, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases}. \tag{2.2}$$

Then $x \in N_\theta^\beta(\Delta^m, I)$ for $\frac{1}{2} < \beta \le 1$ but $x \notin N_\theta^\alpha(\Delta^m, I)$ for $0 < \alpha \le \frac{1}{2}$.
□

Theorem 2.3. *If $x_k \to L(N_\theta^\alpha(\Delta^m, I))$, then $x_k \to L(S_\theta^\alpha(\Delta^m, I))$ and the inclusion is proper.*

Proof. Define a sequence $x=(x_k)$ by

$$\Delta^m x_k = \begin{cases} \left[\sqrt{h_r}\right], & k = 1,2,3,..., \left[\sqrt{h_r}\right] \\ 0, & \text{otherwise} \end{cases}. \tag{2.3}$$

Then we have for every $\varepsilon > 0$ and $\frac{1}{2} < \alpha \le 1$,

$$\frac{1}{h_r^\alpha} \left|\{k \in I_r : |\Delta^m x_k - 0| \ge \varepsilon\}\right| \le \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha},$$

and for any $\delta > 0$ we get

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\{k \in I_r : |\Delta^m x_k - 0| \ge \varepsilon\}\right| \ge \delta \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha} \ge \delta \right\}$$

and so $x_k \to 0\,(S_\theta^\alpha(\Delta^m, I))$ for $\frac{1}{2} < \alpha \le 1$. On the other hand, for $0 < \alpha \le 1$,

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} |\Delta^m x_k - 0| = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r^\alpha} \to \infty$$

and for $\alpha = 1$,

$$\frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r^\alpha} \to 1.$$

Then we can write

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |\Delta^m x_k - 0| \ge 1 \right\} = \left\{ r \in \mathbb{N} : \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r^\alpha} \ge 1 \right\} = \{a, a+1, a+2, ...\} \in F(I)$$

for some $a \in \mathbb{N}$, since I is admissible. Thus $x_k \nrightarrow 0\,(N_\theta^\alpha(\Delta^m, I))$. □

The proof of each of the following results is obvious, so we do not give the proof of theorems.

Theorem 2.4. *If $\liminf_r q_r > 1$, then $x_k \to L(S^\alpha(\Delta^m, I))$ implies $x_k \to L(S_\theta^\alpha(\Delta^m, I))$.*

Theorem 2.5. *If* $\lim_{r\to\infty}\inf \dfrac{h_r^\alpha}{k_r} > 0$, *then* $x_k \to L(S(\Delta^m, I))$ *implies* $x_k \to L(S_\theta^\alpha(\Delta^m, I))$.

Theorem 2.6. *If* $\limsup_r q_r < \infty$, *then* $x_k \to L(S_\theta(\Delta^m, I))$ *implies* $x_k \to L(S(\Delta^m, I))$.

Theorem 2.7. $S_\theta^\alpha(\Delta^m, I) \cap \ell_\infty(\Delta^m)$ *is a closed subset of* $\ell_\infty(\Delta^m)$ *for* $0 < \alpha \le 1$.

Now let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, and let $\alpha, \beta \in (0,1]$ such that $0 < \alpha \le \beta \le 1$. Now we research inclusion connections between the sets of $S_\theta^\alpha(\Delta^m, I)$ –convergent sequences and $N_\theta^\alpha(\Delta^m, I)$ –summable sequences for different $\alpha^{'}s$ and $\theta^{'}s$.

Theorem 2.8. *Let* $\theta = (k_r)$ *and* $\theta' = (s_r)$ *be two lacunary sequences and let* $\alpha, \beta \in (0,1]$ *such that* $0 < \alpha \le \beta \le 1$.

(*i*) *If*

$$\lim_{r\to\infty}\inf \frac{h_r^\alpha}{\ell_r^\beta} > 0 \tag{2.4}$$

then $S_{\theta'}^\beta(\Delta^m, I) \subseteq S_\theta^\alpha(\Delta^m, I)$,

(*ii*) *If*

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^\beta} = 1 \tag{2.5}$$

then $S_\theta^\alpha(\Delta^m, I) \subseteq S_{\theta'}^\beta(\Delta^m, I)$, *where* $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $\ell_r = s_r - s_{r-1}$.

Proof. (*i*) Assume that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let (2.4) be satisfied. For given $\varepsilon > 0$ we have

$$\{k \in J_r : |\Delta^m x_k - L| \ge \varepsilon\} \supseteq \{k \in I_r : |\Delta^m x_k - L| \ge \varepsilon\},$$

$$\frac{1}{\ell_r^\beta}\left|\{k \in J_r : |\Delta^m x_k - L| \ge \varepsilon\}\right| \ge \frac{h_r^\alpha}{\ell_r^\beta}\frac{1}{h_r^\alpha}\left|\{k \in I_r : |\Delta^m x_k - L| \ge \varepsilon\}\right|$$

and so

$$\begin{aligned} &\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k \in I_r : |\Delta^m x_k - L| \ge \varepsilon\}\right| \ge \delta\right\} \\ \subseteq\ &\left\{r \in \mathbb{N} : \frac{1}{\ell_r^\beta}\left|\{k \in J_r : |\Delta^m x_k - L| \ge \varepsilon\}\right| \ge \delta\frac{h_r^\alpha}{\ell_r^\beta}\right\} \in I \end{aligned}$$

for all $r \in \mathbb{N}$. Now taking the limit as $r \to \infty$ in the last inequality and using (2.4) we obtain $S_{\theta'}^\beta(\Delta^m, I) \subseteq S_\theta^\alpha(\Delta^m, I)$.

(ii) Let $x \in S_{\theta}^{\alpha}(\Delta^m, I)$ and (2.5) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^{\beta}}\left|\{k \in J_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| &= \frac{1}{\ell_r^{\beta}}\left|\{s_{r-1} < k \leq k_{r-1} : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\
&\quad + \frac{1}{\ell_r^{\beta}}\left|\{k_r < k \leq s_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\
&\quad + \frac{1}{\ell_r^{\beta}}\left|\{k_{r-1} < k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\
&\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^{\beta}} + \frac{s_r - k_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\
&= \frac{\ell_r - h_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\
&\leq \frac{\ell_r - h_r^{\beta}}{h_r^{\beta}} + \frac{1}{h_r^{\beta}}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\
&\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) + \frac{1}{h_r^{\alpha}}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right|
\end{aligned}$$

and

$$\left\{r \in \mathbb{N} : \frac{1}{\ell_r^{\beta}}\left|\{k \in J_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^{\alpha}}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in I$$

for all $r \in \mathbb{N}$. Thus $S_{\theta}^{\alpha}(\Delta^m, I) \subseteq S_{\theta'}^{\beta}(\Delta^m, I)$. □

Theorem 2.9. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. Then we have*

(i) If (2.4) holds then $N_{\theta'}^{\beta}(\Delta^m, I) \subset N_{\theta}^{\alpha}(\Delta^m, I)$,

(ii) If (2.5) holds and $x \in \ell_{\infty}(\Delta^m)$ then $N_{\theta}^{\alpha}(\Delta^m, I) \subset N_{\theta'}^{\beta}(\Delta^m, I)$.

Proof. Omitted. □

Theorem 2.10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, and let $\alpha, \beta \in (0, 1]$ such that $0 < \alpha \leq \beta \leq 1$. Then*

(i) Let (2.4) holds, if $x_k \to L(N_{\theta'}^{\beta}(\Delta^m, I))$, then $x_k \to L(S_{\theta}^{\alpha}(\Delta^m, I))$,

(ii) Let (2.5) holds and $x = (x_k)$ be a Δ^m-bounded sequence, if $x_k \to L(S_{\theta}^{\alpha}(\Delta^m, I))$, then $x_k \to L(N_{\theta'}^{\beta}(\Delta^m, I))$.

Proof. $i)$ Omitted.

(ii) Assume that $S_\theta^\alpha(\Delta^m, I)-\lim x_k = L$ and $x \in \Delta^m(\ell_\infty)$. Then we may write

$$\begin{aligned}
\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|\Delta^m x_k - L| &= \frac{1}{\ell_r^\beta}\sum_{k\in J_r-I_r}|\Delta^m x_k - L| + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|\Delta^m x_k - L| \\
&\le \left(\frac{\ell_r - h_r}{\ell_r^\beta}\right)M + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|\Delta^m x_k - L| \\
&\le \left(\frac{\ell_r - h_r^\beta}{\ell_r^\beta}\right)M + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|\Delta^m x_k - L| \\
&\le \left(\frac{\ell_r}{h_r^\beta} - 1\right)M + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |x_k - L|\ge\varepsilon}}|\Delta^m x_k - L| + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |x_k - L|<\varepsilon}}|\Delta^m x_k - L| \\
&\le \left(\frac{\ell_r}{h_r^\beta} - 1\right)M + \frac{M}{h_r^\alpha}\left|\{k\in I_r : |\Delta^m x_k - L| \ge \varepsilon\}\right| + \frac{\ell_r}{h_r^\beta}\varepsilon
\end{aligned}$$

and so

$$\left\{r\in\mathbb{N} : \frac{1}{\ell_r^\beta}\sum_{k\in J_r}|\Delta^m x_k - L| \ge \delta\right\} \subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r^\alpha}\left|\{k\in I_r : |\Delta^m x_k - L|\ge\varepsilon\}\right| \ge \frac{\delta}{M}\right\} \in I,$$

for all $r \in \mathbb{N}$. Using (2.5) we obtain that $N_{\theta'}^\beta(\Delta^m, I) - \lim x_k = L$, whenever $S_\theta^\alpha(\Delta^m, I) - \lim x_k = L$. □

References

[1] N. L. Braha, *A new class of sequences related to the ℓ_p spaces defined by sequences of Orlicz functions*, J. Inequal. Appl. 2011, Art. ID 539745, 10 pp.

[2] R. Çolak, *Statistical convergence of order α* Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub, 2010: 121–129.

[3] G. Das, S. K. Mishra, *Banach limits and lacunary strong almost convergence*, J. Orissa Math. Soc. **2** (1983) 61-70.

[4] P. Das, E. Savaş, S. Kr. Ghosal, *On generalizations of certain summability methods using ideals*, Appl. Math. Lett. **24(9)** (2011), 1509–1514.

[5] P. Das, E. Savaş, *On I-statistical and I-lacunary statistical convergence of order α*, Bull. Iranian Math. Soc. **40(2)** (2014), 459–472.

[6] M. Et, H. Altınok, Y. Altın, *On some generalized sequence spaces*, Appl. Math. Comput. **154(1)** (2004), 167–173.

[7] M. Et, *Generalized Cesaro difference sequence spaces of non-absolute type involving lacunary sequences*, Appl. Math. Comput. **219(17)** (2013), 9372-9376.

[8] M. Et, H. Şengül, *Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α*, Filomat **28(8)** (2014), 1593–1602.

[9] M. Et, A. Alotaibi, S. A. Mohiuddine, *On (Δ^m, I)-Statistical Convergence of Order α*, The Scientific World Journal, Volume 2014, Article ID 535419, 5 pages.

[10] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951) 241-244.

[11] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301-313.

[12] A. R. Freedman, J. J. Sember, M. Raphael, *Some Cesaro-type summability spaces*, Proc. Lond. Math. Soc. **37** (1978) 508-520.

[13] J. A. Fridy, C. Orhan, *Lacunary statistical convergence*, Pacific J. Math. **160** (1993) 43-51.

[14] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull. **24(2)** (1981), 169-176.

[15] P. Kostyrko, T. Šalát, W. Wilczyński, *$I-$convergence*, Real Anal. Exchange **26** (2000/2001), 669-686.

[16] P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak, *$I-$convergence and extremal $I-$limit points*, Math. Slovaca **55(4)**(2005), 443-464.

[17] M. Mursaleen, R. Çolak, M. Et, *Some geometric inequalities in a new Banach sequence space*, J. Inequal. Appl. 2007, Art. ID 86757, 6 pp.

[18] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361-375.

[19] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139-150.

[20] T. Šalát, B. C. Tripathy, M. Ziman, *On $I-$convergence field*, Italian J. Pure Appl. Math. **17** (2005), 45–54**.**

[21] E. Savaş, M. Et, *On $\left(\Delta_{\lambda}^{m}, I\right)$-Statistical Convergence of Order α*, Period. Math. Hungar. (2015).

[22] H. Şengül, M. Et, *On lacunary statistical convergence of order α*, Acta Math. Sci. Ser. B Engl Ed (2014), **34(2)**:473–482.

[23] B. C. Tripathy, *Matrix transformation between some classes of sequences*, J. Math. Analysis and Appl. **206 (2)** (1997), 448-450.

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